The Existence and Uniqueness of the Taylor Series of Iterated Functions

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Abstract

The Ackermann function is comprised of addition, multiplication, exponentiation, tetration, pentation and so on, formed from an infinite series of arithmetic operators or hyperoperators. The hyperoperators beyond addition are recursively defined from their predecessor using iterated functions. Addition, multiplication and exponentiation are defined for complex numbers, but the higher hyperoperators beginning with tetration are only defined for the whole numbers.

Fractional iteration addresses questions like, "Do maps have flows?" and "Can a function be iterated by a complex value?" The existence and uniqueness of the Taylors series of iterated functions is proven. In the general case, the Taylors series of iterated functions can only be iterated by whole numbers; it is discrete in time. But in cases consistent with the classification of fixed points as well as both Schroeder's equation and Abel's equation, complex iterates are defined. This in turn uniquely extends the hyperoperators like tetration to be defined for complex values.

1 Introduction

The branch of mathematics known as dynamics concerns itself with mathematical systems that evolve over time. Iterated functions[1, 2, 3, 4] are such a system and are important in both arithmetic[7] and mathematical physics[1, 5], where they provide an alternate mathematical model of physics to that of partial differential equations.

In arithmetic the Ackermann function and the hyperoperators are defined using iterated functions[7]. There is considerable ongoing interest in extending tetration from the whole numbers to the real and complex numbers[9, 10]. This extension can happen by the use of iterated functions in the definition of hyperoperators. When maps have flows the iterator is extended from the whole numbers to the real numbers. Extending the iterator from the whole numbers to the complex numbers allows the hyperoperators of tetration and beyond to be defined for complex values.

Fractional iteration[8] can be defined using either Bell[1] and Carleman[3] infinite matrices, taking advantage of the fact that the composition of functions can be performed by matrix multiplication. Once a matrix has been diagonalized, fractional iterates are found by raising the matrix to the appropriate power.

This paper establishes the existence and uniqueness of the Taylor series of iterated functions as a tool to help solve iterated functional equations and to extend tetration and the hyperoperations. The Taylor series of iterated functions advantages over ealier approaches are that it is more general, encompassing both Schröder's and Abel's equation, and it provides more combinatorial insight into iterated functions.

Iterated functional equations focus on questions like what is the functional "square root" of a function $f(z)$, solving for $g(x)$ where $g(g(x)) = f(x)$.

Consider the holomorphic function $f(z) : \mathbb{C} \to \mathbb{C}$ with a fixed point at f_0 and its iterates $f^t(z), t \in \mathbb{N}$. This gives $f(z) = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^n$ for $0 \leq |z| < R$ for some positive R. Note that $f(z)$ is the exponential generating function of the sequence $f_0, f_1, \ldots, f_\infty$, where $f(f_0) = f_0$ and f_1 will be written as λ. The expression f_j^k denotes $(D^j f(z))^k|_{z=f_0}$.

The symbol t for time assumes $t \in \mathbb{N}$, that time is discrete. This allows the variable n to be used solely in the context of differentiation in this paper. Beginning with the second derivative each component will be expressed in a general form using summations.

2 The Derivatives of Iterated Functions

2.1 The First Derivative

The first derivative of a function at its fixed point $Df(f_0) \equiv \lambda$ is referred to as the multiplier or the Lyapunov characteristic number; its logarithm is known as the Lyapunov exponent. Let $g(z) = f^{t-1}(z)$, then

$$\begin{aligned} Df(g(z)) &= f'(g(z))g'(z) \\ &= f'(f^{t-1}(z))Df^{t-1}(z) \\ &= \prod_{k_1=0}^{t-1} f'(f^{t-k_1-1}(z)) \end{aligned}$$

$$\begin{aligned} Df^t(f_0) &= f'(f_0)^t \\ &= f_1^t = \lambda^t \end{aligned} \quad (1)$$

2.2 The Second Derivative

$$\begin{aligned} D^2 f(g(z)) &= f''(g(z))g'(z)^2 + f'(g(z))g''(z) \\ &= f''(f^{t-1}(z))(Df^{t-1}(z))^2 + f'(f^{t-1}(z))D^2 f^{t-1}(z) \end{aligned}$$

Setting $g(z) = f^{t-1}(z)$ results in

$$D^2 f^t(f_0) = f_2 \lambda^{2t-2} + \lambda D^2 f^{t-1}(f_0)$$

When $\lambda \neq 0$, a recurrence equation is formed that is solved as a summation.

$$\begin{aligned} D^2 f^t(f_0) &= f_2 \lambda^{2t-2} + \lambda D^2 f^{t-1}(f_0) \\ &= \lambda^0 f_2 \lambda^{2t-2} \end{aligned}$$

$$
\begin{aligned}
& +\lambda^1 f_2 \lambda^{2t-4} \\
& +\cdots \\
& +\lambda^{t-2} f_2 \lambda^2 \\
& +\lambda^{t-1} f_2 \lambda^0 \\
= \; & f_2 \sum_{k_1=0}^{t-1} \lambda^{2t-k_1-2} \qquad (2)
\end{aligned}
$$

2.3 The Third Derivative

Continuing on with the third derivative

$$
\begin{aligned}
D^3 f(g(z)) &= f'''(g(z))g'(z)^3 + 3f''(g(z))g'(z)g''(z) + f'(g(z))g'''(z) \\
&= f'''(f^{t-1}(z))(Df^{t-1}(z))^3 \\
&\quad +3f''(f^{t-1}(z))Df^{t-1}(z)D^2 f^{t-1}(z) \\
&\quad +f'(f^{t-1}(z))D^3 f^{t-1}(z)
\end{aligned}
$$

$$
\begin{aligned}
D^3 f^t(f_0) &= f_3 \lambda^{3t-3} + 3f_2^2 \sum_{k_1=0}^{t-1} \lambda^{3t-k_1-5} + \lambda D^3 f^{t-1}(f_0) \\
&= f_3 \sum_{k_1=0}^{t-1} \lambda^{3t-2k_1-3} + 3f_2^2 \sum_{k_1=0}^{t-1} \sum_{k_2=0}^{t-k_1-2} \lambda^{3t-2k_1-k_2-5} \qquad (3)
\end{aligned}
$$

Note that the index k_1 from the second derivative, is renamed k_2 in the final summation of the third derivative. A certain amount of renumbering is unavoidable in order to use a simple index scheme.

2.4 The n^{th} Derivative

Let $f(z)$ and $g(z)$ be holomorphic functions, then the Bell polynomials can be constructed using Faa Di Bruno's formula.

$$
D^n f(g(z)) = \sum_{\pi(n)} \frac{n!}{k_1! \cdots k_n!} (D^k f)(g(z)) \left(\frac{Dg(z)}{1!}\right)^{k_1} \cdots \left(\frac{D^n g(z)}{n!}\right)^{k_n}
$$

A partition of n is $\pi(n)$, usually denoted by $1^{k_1} 2^{k_2} \cdots n^{k_n}$ with $k_1 + 2k_2 + \cdots nk_n = k$; where k_i is the number of parts of size i. The partition function $p(n)$ is a decategorized version of $\pi(n)$; the function $\pi(n)$ enumerates the integer partitions of n, while $p(n)$ is the cardinality of the enumeration of $\pi(n)$.

Setting $g(z) = f^{t-1}(z)$ results in

$$
D^n f^t(z) =
$$

$$
\textstyle\sum_{\pi(n)} \frac{n!}{k_1! \cdots k_n!} (D^k f)(f^{t-1}(z)) \left(\frac{Df^{t-1}(z)}{1!}\right)^{k_1} \cdots \left(\frac{D^n f^{t-1}(z)}{n!}\right)^{k_n}
$$

The Taylor series of $f^t(z)$ is derived by evaluating the derivatives of the iterated function at a fixed point $f^t(f_0)$ by setting $z = 0$ and separating out the k_n term of the summation that is dependent on $D^n f^{t-1}(f_0)$.

$$D^n f^t(f_0) = \sum \tfrac{n!(D^k f)(f_0)}{k_1! \cdots k_{n-1}!} \left(\tfrac{Df^{t-1}(f_0)}{1!}\right)^{k_1} \cdots \left(\tfrac{D^n f^{t-1}(f_0)}{(n-1)!}\right)^{k_{n-1}} + (Df)(f_0) D^n f^{t-1}(f_0)$$

and rewriting $(D^k f)(f_0)$ as f_k.

$$D^n f^t(f_0) = \sum \tfrac{n! f_k}{k_1! \cdots k_{n-1}!} \left(\tfrac{Df^{t-1}(f_0)}{1!}\right)^{k_1} \cdots \left(\tfrac{D^{n-1} f^{t-1}(f_0)}{(n-1)!}\right)^{k_{n-1}} + \lambda D^n f^{t-1}(f_0)$$

The remaining $p(n) - 1$ terms of the summation are only dependent on $D^k f^{t-1}(f_0)$, where $0 < k < n$. Let this partial summation be written as $\sigma(n)$ with $\sigma(0) = 0$ and $\sigma(1) = 1$.

$$\sigma(n) = \sum \frac{n! f_k}{k_1! \cdots k_{n-1}!} \left(\frac{Df^{t-1}(f_0)}{1!}\right)^{k_1} \cdots \left(\frac{D^{n-1} f^{t-1}(f_0)}{(n-1)!}\right)^{k_{n-1}} \tag{4}$$

Rewriting the $p(n) - 1$ terms of the summation as $\sigma(n)$ will help in writing a proof by general induction. For $n > 1$

$$D^n f^t(f_0) = \sigma(n) + \lambda D^n f^{t-1}(f_0) \tag{5}$$

2.5 Existence and Uniqueness of Iterated Functions

Theorem 1 *The Taylor series of an iterated holomorphic function $f^t(z)$ can be constructed given a fixed point where $t \in \mathbb{N}$.*

Proof. Assume the given fixed point is at zero. The Taylor series of $f^t(z)$ can be constructed for some positive value of R where $0 < |z| < R$ if and only if $D^n f^t(f_0)$ can be constructed for every $n \geq 0$. Prove by strong induction.

Basis Steps:

Case $n = 0$. By definition $D^0 f^t(f_0) = f_0$, so $D^0 f^t(f_0)$ can be constructed.

Case $n = 1$. From Eq. 1, $D^1 f^t(f_0) = \lambda^t$, so $D^1 f^t(f_0)$ can be constructed.

Induction Step:

Case. Assume that $D^k f^t(f_0)$ can be constructed for all k, where $0 \leq k < n$. Using Eq.5, $D^k f^t(f_0) = \sigma(k) + \lambda D^k f^{t-1}(f_0)$. The function $\sigma(k)$ in only dependent on $D^0 f(f_0), \ldots, D^k f(f_0)$ and $D^k f^t(f_0), \ldots, D^{(k-1)} f^t(f_0)$. By the strong induction hypothesis, $\sigma(k)$ can be constructed. Therefore Eq.5 can be reduced to a geometrical progression based on λ that can be represented by a summation.

$$D^k f^t(f_0) = \sum_{j=0}^{k-1} \sigma(k) \lambda^j \tag{6}$$

This completes the induction step that $D^n f^t(f_0)$ can be constructed for all whole numbers n.

The Taylor series for $f^t(z)$ is

$$f^t(z) = \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \frac{\sigma(n)}{n!} \lambda^j (z - f_0)^n \tag{7}$$

■

Theorem 2 *The Existence and Uniqueness of the Taylor Series of Iterated Functions.*

Proof. The construction of the the Taylor series of iterated functions proves its existence, while the uniqueness of Taylor series proves its uniqueness.

■

Theorem 3 *The Taylor Series of Iterated Holomorphic Functions are Holomorphic.*

Proof. Compositions of holomorphic functions are holomorphic.

■

2.6 Classification of Fixed Points and Linearization

Define θ such that $\theta \in \mathbb{R}$ where $\lambda = e^{2\pi i\theta}$. If θ is "badly approximable"[4] to a rational number, then θ is Diophantine. Most real numbers are Diophantine.

Note that while we have stated our interest in fractional iteration, Eq.7 has not simplified the summations. The reason is seen upon investigating the simpler summation of geometrical progression $\sum_{j=0}^{k-1} \lambda^j$. The standard simplification of the geometrical progression doesn't hold where λ is a root of unity. The simplifications are handled on a case by case basis, likewise the fractional iteration or linearization can't be said to exist in the general sense. But linearization does exist most cases.

The historical approach[2] to continuous iterates of functions is to use Schroeder's equation where $|\lambda| \neq 1$ or θ is Diophantine and Abel's equation where $\lambda = 1$. Eq.5 is consistent with both approaches as well as the Classification of Fixed Points[4].

2.6.1 Classification of Fixed Points in the Complex Plane

- *Superattracting*: $\lambda = 0$
- *Hyperbolic*: $|\lambda| \neq 1$
  - *Attracting*: $|\lambda| < 1$
  - *Repelling*: $|\lambda| > 1$
- *Parabolic Rationally Neutral*: $\lambda = 1$
- *Rationally Neutral*: $|\lambda| = 1$ and $\lambda^k = 1$
- *Irrationally Neutral*: $|\lambda| = 1$ and $\lambda^k \neq 1$

– *Diophantine*: θ is Diophantine.

– *non-Diophantine*: θ is not Diophantine.

The term Diophantine will be used for irrationally neutral fixed points where θ is Diophantine and the term non-Diophantine will be used for irrationally neutral fixed points where θ is not Diophantine.

2.6.2 Linearization

Theorem 4 ***Complex Linearization Theorem*** *The Taylor series of an iterated holomorphic function $f^t(z)$ can be constructed given a non-superattracting fixed point that is not non-Diophantine and where $t \in \mathbb{C}$.*

Proof. By case.

Case λ *not a root of unity or is Diophantine*, Schröder's equation.

$$\begin{aligned} f^t(z) &= f_0 + \lambda^t(z - f_0) + \frac{1}{2} f_2 \frac{\lambda^{2t} - \lambda^t}{\lambda^2 - \lambda}(z - f_0)^2 \\ &+ \frac{1}{6}(f_3 \frac{\lambda^{3t} - \lambda^t}{\lambda^3 - \lambda^2} + 3f_2^2 \frac{\lambda^{t-2}(\lambda^t - 1)(\lambda^t - \lambda)}{(\lambda - 1)^2(\lambda + 1)})(z - f_0)^3 \\ &+ \ldots \end{aligned} \quad (8)$$

Case $\lambda = 1$, Abel's equation.

$$f^t(z) = z + \frac{1}{2} t f_2 (z - f_0)^2 + \frac{1}{12}(3(t^2 - t)f_2^2 + 2tf_3)(z - f_0)^3 + \ldots \quad (9)$$

Case $\lambda^k = 1$ *with* $\tau = t/k$.

$$f^t(z) = z + \frac{1}{2} \tau f_2 (z - f_0)^2 + \frac{1}{12}(3(\tau^2 - \tau)f_2^2 + 2\tau f_3)(z - f_0)^3 + \ldots \quad (10)$$

■

Theorem 5 ***The Existence and Uniqueness of the Linearization of Iterated Functions*** *The linearization of an iterated holomorphic function $f^t(z)$ exists and is unique given a non-superattracting fixed point that is not non-Diophantine and where $t \in \mathbb{C}$.*

Proof. By Theorem 2 and Theorem 4 there are three cases to consider - λ not a root of unity or is Diophantine, $\lambda = 1$, and $\lambda^k = 1$ with $\tau = t/k$. In each case a linearization can be constructed, thus the linearization exists. Once again consider the three cases, each representing a different topological conjugacy. But in each of the three cases, the linearization represents all posible homomorphic functions for that topological conjugacy Eq. 8, 9 10. The topological conjugacy imposes the uniqueness of the three cases, but each case constructs all posible solutions for each case. Therefore it is unique, ignoring specific examples of the general case. So the linearization of an iterated holomorphic function $f^t(z)$ is proven to exists and be unique upto the fixed point.

■

2.6.3 Iterated Sine Function

An example of linearization is the $\sin(z)$ function that has a fixed point at zero and it's first derivitive $\lambda = 1$.

$$\begin{aligned}\sin^t(z) &= z - \frac{t}{6}z^3 + \left(\frac{t^2}{24} - \frac{t}{30}\right)z^5 + \left(-\frac{5t^3}{432} + \frac{t^2}{45} - \frac{41t}{3780}\right)z^7 \\ &+ \left(\frac{35t^4}{10368} - \frac{71t^3}{6480} + \frac{67t^2}{5670} - \frac{4t}{945}\right)z^9 + \dots\end{aligned}$$

3 Extending The Hyperoperators

Let $a \uparrow^m b$ be the Conway chained arrow notation for the m^{th} hyperoperator after multiplication. Then $a \uparrow b \equiv a^b$ is exponentiation and $a \uparrow^2 b \equiv {}^b a$ is tetration and so on.

3.1 Extending Tetration

In extending tetration to the complex numbers by linearization it is useful to note the locii of a for ${}^z a$ where $\lambda = e^{2\pi i\theta}$. The locii of a will be referred to as the tetration boundary of convergence since its interior has the condition that $|\lambda| < 1$.

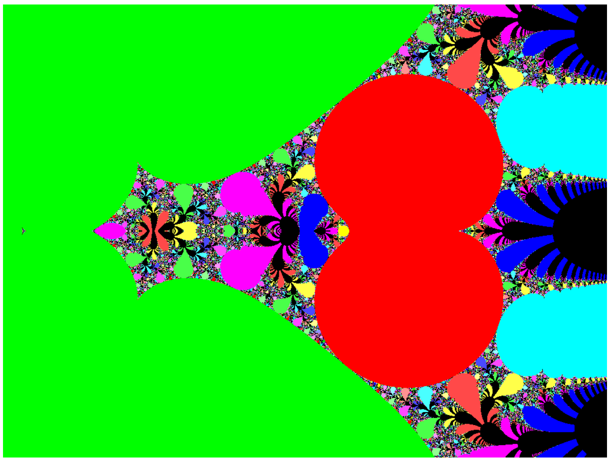

Figure 1: Tetration Mandelbrot Set by Period

The large red kidney shaped area, in Fig. 3.1, is the tetration area of convergence which is period one. The black areas show where the exponential map escapes to infinity.

Theorem 6 ***The Tetration Boundary of Convergence*** *The tetration boundary of convergence is* $a = e^{e^{2i\pi\theta - e^{2i\pi\theta}}}$ *where* $\theta \in \mathbb{R}$ *and* $0 \leq \theta \leq 1$.

Proof.

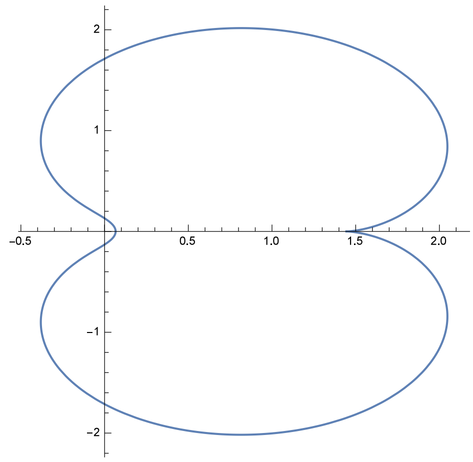

Figure 2: Tetration Boundary of Convergence

Let $a^z = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ with a_0 as a fixed point for a^z. The first derivitive evaluated at a fixed point gives λ, therefore

$$Da^z = De^{log(a)z} = log(a)a^z$$

and

$$\lambda = log(a)a^z|_{z=a_0} = log(a)a^{a_0} = log(a)a_0 = log(a^{a_0}) = log(a_0).$$

Since $\lambda = e^{2\pi i\theta} = \log(a_0)$, we have $a_o = e^{e^{2\pi i\theta}}$. The definition of an exponential map's fixed point $a_0 = a^{a_0}$ gives $a = a_0^{\frac{1}{a_0}}$. Thus

$$a = e^{e^{2i\pi\theta - e^{2i\pi\theta}}}.$$

■

3.2 Existence and Uniqueness of Hyperoperators

Theorem 7 *Assume $a \uparrow^k z$ has a non-superattracting fixed point and is not non-Diophantine, where $0 < k < m$ and $k, m \in \mathbb{Z}^+$. Then the hyperoperator $a \uparrow^m b$ is defined, where $a, b \in \mathbb{C}$ and can be constructed.*

Proof. By strong induction.

Basis Step:

Case $k = 1$. Given $a \uparrow^k b = a \uparrow b$ is defined, holomorphic and has a non-superattracting fixed point and is not non-Diophantine, let $f(z) \equiv a \uparrow z$, then $a \uparrow^2 b \equiv f^b(1)$. Thus $a \uparrow^2 b$ is defined.

Induction Step:

Case $k = i - 1$. Given $a \uparrow^1 b$ through $a \uparrow^{i-1} b$ are defined, holomorphic and have a non-superattracting fixed point and are not non-Diophantine, let $f(z) \equiv a \uparrow^{i-1} z$, then $a \uparrow^i b \equiv f^b(1)$. Thus $a \uparrow^i b$ is defined.

This completes the induction step and the proof.

■

Theorem 8 *Assume $a \uparrow^k z$ has a non-superattracting fixed point and is not non-Diophantine, where $0 < k \leq m$ and $k, m \in \mathbb{Z}^+$. Given that the hyperoperator $a \uparrow^m b$ is defined, where $a, b \in \mathbb{C}$, then it is unique up to a fixed point.*

Proof. Let $f(z) \equiv a \uparrow^{m-1} z$, then $a \uparrow^m b \equiv f^b(1)$. Since $f(z)$ has a non-superattracting fixed point and is not non-Diophantine, Theorem 2 is true proving the hyperoperator $a \uparrow^m b$ is unique up to a fixed point.
■

Just as the logarithm is multivalued, the different fixed points of tetration and the higher hyperoperations result in them being multivalued. The inability to linearize values of a that are non-Diophantine, combined with the collection of fixed points for each $a \uparrow^k z$, where $0 < k \leq m$ and $k, m \in \mathbb{Z}^+$.

Theorem 9 *The inverse function of $a \uparrow^m z$ can be constructed.*

Proof. Let $f(z) = a \uparrow^m z$, then from Theorem 4, $f^{-1}(z)$ can be constructed.
■

Below is an example of how the hyperoperators can be extended to the real numbers where the fixed points are all real:

$$\begin{aligned}
\sqrt{2} \uparrow^2 \infty &= 2 \\
\sqrt{2} \uparrow^3 \infty &= 1.54912 \\
\sqrt{2} \uparrow^4 \infty &= 1.48436 \\
\sqrt{2} \uparrow^5 \infty &= 1.45915 \\
\sqrt{2} \uparrow^6 \infty &= 1.44615 \qquad (11)
\end{aligned}$$

4 Concluding Remarks

The Taylor series of iterated functions as derived only requires that the function to be iterated is holomorphic and has a fixed point. While there are important cases where the iterator can take on real and complex values, if there is a completely general case where maps can be proven to have flows, it is beyond the scope of the mathematics presented in this paper. Also, the scope of the paper is limited to the iteration of complex functions; future research may be able to extend the results presented here to more general functions.

Future research may also investigate the combinatorial basis of iterated functions. Just as set partitions form the combinatorial basis of composite functions, preliminary research indicates that total partitions[6], also known as Schröder's Fourth Problem, form the combinatorial basis of iterated functions. Mathematica code has been written that evaluates the individual total partitions for their associated analytic expression, but the code only works for the first six derivatives of an iterated function.

A major issue regarding extending tetration is the uniqueness of the solution. A stock answer about extending tetration, given on math websites, is that there are a number of proposed solutions. The solution presented here for extending tetration and the hyperoperators is guaranteed to be consistent with the values of tetration and the hyperoperators for the natural numbers. Along with the

Theorem 4 these two conditions uniquely extend tetration and the hyperoperators to the complex numbers. Experimental results appear to be consistent with the results from [1].

References

[1] R. Aldrovandi and L. P. Freitas. Continuous iteration of dynamical maps. *J. Math. Phys.*, 39(10):5324–5336, 1998.

[2] Daniel S. Alexander. *A history of complex dynamics. From Schröder to Fatou and Julia.* Braunschweig: Vieweg, 1994.

[3] G. Berkolaiko, S. Rabinovich, and S. Havlin. Analysis of Carleman Representation of Analytical Recursions. *J. Math. Anal. Appl.*, 224:81–90, 1998.

[4] L. Carleson and T. Gamelin. *Complex Dynamics.* Universitext / Universitext: Tracts in Mathematics, 1993.

[5] E. Atlee Jackson. *Perspectives of nonlinear dynamics, Vol. 1.* Cambridge University Press, 1989.

[6] R. P. Stanley. *Enumerative Combinatorics, Vol. 2.* Cambridge, 1999.

[7] Jean van Heijenoort. *From Frege to Godel.* Harvard University Press, 1967.

[8] MathOverflow website. Fractional iteration tag. Published electronically at http://mathoverflow.net/questions/tagged/fractional-iteration, 2016.

[9] MathStackexchange website. Hyperoperation tag. Published electronically at http://math.stackexchange.com/questions/tagged/hyperoperation, 2016.

[10] MathStackexchange website. Tetration tag. Published electronically at http://math.stackexchange.com/questions/tagged/tetration, 2016.

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