Bell Polynomials of Iterated Functions

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Abstract

Physics can be modeled by PDEs or recursively as iterated functions. A method is presented to decompose iterated functions into Bell polynomials and then into the combinatoric structure Schroeder's Fourth Problem. Consistency with complex dynamics is shown by deriving the classification of fixed points in complex dynamics.

1 Introduction

Stephen Wolfram pointed out in 1986 [7] that the problem of extending tetration to the complex numbers was actually part of the much larger and more important problem of unifying the discreet representation of chaotic systems in mathematics with the continuous representation of chaotic systems in physics, of unifying maps from iterated functions and flows from PDEs. He maintained that the duality prevented the derivation of mathematical solutions for continuous chaotic systems as are found in physics. Wolfram suggested at if tetration could be defined for complex numbers then those results might be generalized to unify discrete maps and continuous flows.

R. Aldrovandi and L. P. Freitas published *Continuous iteration of dynamical maps*[1] where the iterated functions of hyperbolic dynamical systems were decomposed into Bell matrices. The composition of functions then becomes the multiplication of matices giving continuous iteration by taking the power of diagonalized matices. Finally a simplified version of the Navier Stokes equation is analyzed.

This paper dispenses with the need of matrices by constructing the Taylors series of an iterated function from Bell polynomials instead of Bell matrices. Bell polynomials are the derivatives of composite functions and are given by Faa Di Bruno's formula[3][9]. Bell numbers or set partitions describe the underlying combinatorics behind Bell polynomials. The Bell polynomials in turn are indexed by the combinatorial structure total partitions[12]. This allows iterated functions to be constructed by enumerating and then evaluating the total partitions, recovering the Bell polynomials. Not only is the restriction to hyperbolic dynamical systems removed but consistency with complex dynamics is shown by deriving the classification of fixed points.

2 The Derivatives of Iterated Functions

Consider the holomorphic function $f(z) : \mathbb{C} \to \mathbb{C}$ and its iterates $f^t(z), t \in \mathbb{N}$. The standard convention of using a coordinate translation to set a fixed point at zero is invoked, $f(0) \equiv 0$, giving $f(z) = \sum_{n=1}^{\infty} \frac{f_n}{n!} z^n$ for $0 \le |z| < R$ for some positive R. Note that $f(z)$ is the exponential generating function of the sequence $f_0, f_1, \ldots, f_\infty$, where $f_0 = 0$ and f_1 will be written as λ. The expression f_j^k denotes $(D^j f(z))^k|_{z=0}$.

Note: The symbol t for time assumes $t \in \mathbb{N}$, that time is discrete. This allows the variable n to be used solely in the context of differentiation in this paper. Beginning with the second derivative each component will be expressed in a general form using summations and referred to here as *Schroeder summations.*

2.1 The First Derivative

The first derivative of a function at its fixed point $Df(0) = f_1$ is often represented by λ and referred to as the multiplier or the Lyapunov characteristic number; its logarithm is known as the Lyapunov exponent. Let $g(z) = f^{t-1}(z)$, then

$$\begin{aligned} Df(g(z)) &= f'(g(z))g'(z) \\ &= f'(f^{t-1}(z))Df^{t-1}(z) \\ &= \prod_{k_1=0}^{t-1} f'(f^{t-k_1-1}(z)) \end{aligned}$$

$$\begin{aligned} Df^t(0) &= f'(0)^t \\ &= f_1^t = \lambda^t \end{aligned} \tag{1}$$

2.2 The Second Derivative

$$\begin{aligned} D^2 f(g(z)) &= f''(g(z))g'(z)^2 + f'(g(z))g''(z) \\ &= f''(f^{t-1}(z))(Df^{t-1}(z))^2 + f'(f^{t-1}(z))D^2 f^{t-1}(z) \end{aligned}$$

Setting $g(z) = f^{t-1}(z)$ results in

$$D^2 f^t(0) = f_2 \lambda^{2t-2} + \lambda D^2 f^{t-1}(0)$$

When $\lambda \neq 0$, a recurrence equation is formed that is solved as a summation.

$$\begin{aligned} D^2 f^t(0) &= f_2 \lambda^{2t-2} + \lambda D^2 f^{t-1}(0) \\ &= \lambda^0 f_2 \lambda^{2t-2} \\ &\quad + \lambda^1 f_2 \lambda^{2t-4} \\ &\quad + \cdots \end{aligned}$$

$$
\begin{aligned}
&+\lambda^{t-2} f_2 \lambda^2 \\
&+\lambda^{t-1} f_2 \lambda^0 \\
= \;& f_2 \sum_{k_1=0}^{t-1} \lambda^{2t-k_1-2} \qquad (2)
\end{aligned}
$$

2.3 The Third Derivative

Continuing on with the third derivative,

$$
\begin{aligned}
D^3 f(g(z)) &= f'''(g(z))g'(z)^3 + 3f''(g(z))g'(z)g''(z) + f'(g(z))g'''(z) \\
&= f'''(f^{t-1}(z))(Df^{t-1}(z))^3 \\
&\quad +3f''(f^{t-1}(z))Df^{t-1}(z)D^2 f^{t-1}(z) \\
&\quad +f'(f^{t-1}(z))D^3 f^{t-1}(z)
\end{aligned}
$$

$$
\begin{aligned}
D^3 f^t(0) &= f_3 \lambda^{3t-3} + 3f_2^2 \sum_{k_1=0}^{t-1} \lambda^{3t-k_1-5} + \lambda D^3 f^{t-1}(0) \\
&= f_3 \sum_{k_1=0}^{t-1} \lambda^{3t-2k_1-3} + 3f_2^2 \sum_{k_1=0}^{t-1} \sum_{k_2=0}^{t-k_1-2} \lambda^{3t-2k_1-k_2-5} \qquad (3)
\end{aligned}
$$

Note that the index k_1 from the second derivative is renamed k_2 in the final summation of the third derivative. A certain amount of renumbering is unavoidable in order to use a simple index scheme.

2.4 The Fourth Derivative

Now the fourth derivative,

$$
\begin{aligned}
D^4 f(g(z)) &= f^{(4)}(g(z))g'(z)^4 + 3f''(g(z))g''(z)^2 + f'(g(z))g^{(4)}(z) \\
&\quad +4g'(z)f''(g(z))g'''(z) + 6g'(z)^2 f'''(g(z))g''(z) \\
&= f^{(4)}(f^{t-1}(z))(Df^{t-1}(z))^4 \\
&\quad +3f''(f^{t-1}(z))(D^2 f^{t-1}(z))^2 \\
&\quad +4Df^{t-1}(z)f''(f^{t-1}(z))(D^3 f^{t-1}(z)) \\
&\quad +6(Df^{t-1}(z))^2 f'''(f^{t-1}(z))(D^2 f^{t-1}(z)) \\
&\quad +f'(f^{t-1}(z))D^4 f^{t-1}(z)
\end{aligned}
$$

$$
D^4 f^t(0) = 12 f_2^3 \sum_{k_1=0}^{t-1} \sum_{k_2=0}^{t-k_1-2} \sum_{k_3=0}^{t-k_1-k_2-3} \lambda^{4\,t-3\,k_1-2\,k_2-k_3-9} \qquad (4)
$$

$$+4f_2f_3\sum_{k_1=0}^{t-1}\sum_{k_2=0}^{t-k_1-2}\lambda^{4\,t-3\,k_1-2\,k_2-7}$$

$$+3f_2^3\sum_{k_1=0}^{t-1}\sum_{k_2=0}^{t-k_1-2}\sum_{k_3=0}^{t-k_1-2}\lambda^{4\,t-3\,k_1-k_2-k_3-8}$$

$$+6f_2f_3\sum_{k_1=0}^{t-1}\sum_{k_2=0}^{t-k_1-2}\lambda^{4\,t-3\,k_1-k_2-6}$$

$$+f_4\sum_{k_1=0}^{t-1}\lambda^{4\,t-k_1-4}$$

2.5 The n^{th} Derivative

Let $f(z)$ and $g(z)$ be holomorphic functions, then the Bell polynomials can be constructed using Faa Di Bruno's formula. [9]

$$D^n f(g(z)) = \sum_{\pi(n)} \frac{n!}{k_1!\cdots k_n!}(D^k f)(g(z))\left(\frac{Dg(z)}{1!}\right)^{k_1} \cdots \left(\frac{D^n g(z)}{n!}\right)^{k_n} \quad (5)$$

A partition of n is $\pi(n)$, usually denoted by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1 + 2k_2 + \cdots nk_n = k$; where k_i is the number of parts of size i. The partition function $p(n)$ is a decategorized version of $\pi(n)$, the function $\pi(n)$ enumerates the integer partitions of n, while $p(n)$ is the cardinality of the enumeration of $\pi(n)$.

Setting $g(z) = f^{t-1}(z)$ results in

$$D^n f^t(z) = \sum_{\pi(n)} \frac{n!}{k_1!\cdots k_n!}(D^k f)(f^{t-1}(z))\left(\frac{Df^{t-1}(z)}{1!}\right)^{k_1} \cdots \left(\frac{D^n f^{t-1}(z)}{n!}\right)^{k_n} \quad (6)$$

The Taylors series of $f^t(z)$ is derived by evaluating the derivatives of the iterated function at a fixed point $f^t(0)$ by setting $z = 0$ and separating out the k_n term of the summation that is dependent on $D^n f^{t-1}(0)$.

$$D^n f^t(0) = \sum \frac{n!(D^k f)(0)}{k_1!\cdots k_{n-1}!}\left(\frac{Df^{t-1}(0)}{1!}\right)^{k_1} \cdots \left(\frac{D^n f^{t-1}(0)}{(n-1)!}\right)^{k_{n-1}} + (Df)(0)D^n f^{t-1}(0) \quad (7)$$

and rewriting $(D^k f)(0)$ as f_k

$$\begin{aligned} D^n f^t(0) &= \sum \frac{n! f_k}{k_1! \cdots k_{n-1}!} \left(\frac{Df^{t-1}(0)}{1!} \right)^{k_1} \cdots \left(\frac{D^{n-1} f^{t-1}(0)}{(n-1)!} \right)^{k_{n-1}} \\ & \quad + \lambda D^n f^{t-1}(0) \end{aligned} \tag{8}$$

The remaining $p(n) - 1$ terms of the summation are only dependent on $D^k f^{t-1}(0)$, where $0 < k < n$. Let this partial summation be written as $\sigma(n)$ with $\sigma(0) = 0$ and $\sigma(1) = 1$.

$$\sigma(n) = \sum \frac{n! f_k}{k_1! \cdots k_{n-1}!} \left(\frac{Df^{t-1}(0)}{1!} \right)^{k_1} \cdots \left(\frac{D^{n-1} f^{t-1}(0)}{(n-1)!} \right)^{k_{n-1}} \tag{9}$$

Rewriting the $p(n) - 1$ terms of the summation as $\sigma(n)$ will help in writing a proof by general induction. For $n > 1$,

$$D^n f^t(0) = \sigma(n) + \lambda D^n f^{t-1}(0) \tag{10}$$

Theorem 1 *The Taylor series of an iterated holomorphic function $f^t(z)$ can be constructed given a fixed point where $t \in \mathbb{N}$.*

Proof. Assume the given fixed point is at zero. The Taylor series of $f^t(z)$ can be constructed for some positive value of R where $0 < |z| < R$ if and only if $D^n f^t(0)$ can be constructed for every $n \geq 0$. prove by strong induction.

Basis Steps:

Case $n = 0$. By definition $D^0 f^t(0) = 0$, so $D^0 f^t(0)$ can be constructed.

Case $n = 1$. From Eq. 1, $D^1 f^t(0) = \lambda^t$, so $D^1 f^t(0)$ can be constructed.

Induction Step: Case $n = k$. Assume that $D^k f^t(0)$ can be constructed for all k where $0 \leq k < n$. Using Eq. 10, $D^k f^t(0) = \sigma(k) + \lambda D^k f^{t-1}(0)$. The function $\sigma(k)$ in only dependent on $D^0 f(0), \ldots, D^k f(0)$ and $D^k f^t(0), \ldots, D^{(k-1)} f^t(0)$. By the strong induction hypothesis, $\sigma(k)$ can be constructed. Therefore Eq. 10 can be reduced to a geometrical progression based on λ that can be represented by a summation.

$$D^k f^t(0) = \sum_{j=0}^{k-1} \sigma(k) \lambda^j \tag{11}$$

This completes the induction step that $D^n f^t(0)$ can be constructed for all whole numbers n.

The Taylors series for $f^t(z)$ is

$$f^t(z) = \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \frac{\sigma(n)}{n!} \lambda^j z^n \tag{12}$$

■

3 Combinatorics of Iterated Functions

The sum of the numerical coefficients from Equation 1 is 1, Equation 2 is 1, Equation 3 is $3 + 1 = 4$ and Equation 5 is $12 + 4 + 3 + 6 + 1 = 26$ give the first four terms of EIS A000311[4], Schroeder's Fourth Problem or labeled hierarchies. On page 197 of Riordan's Combinatorial Identities this combinatoric structure is listed as the structure associated with Bell polynomials.

Items	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Integer Partitions	1	2	3	5	7	11	15	22
Set Partitions	1	2	5	15	52	203	877	4140
Unlabeled Hierarchies	1	1	2	5	12	33	90	261
Total Partitions	1	1	4	26	236	2752	39208	660032

Table 1: Several Combinatoric Structures

Likewise, the number of additive terms are related to EIS A000669[4], unlabeled hierarchies[5] or series-reduced planted trees. This similar to the manner in which EIS A000041 – integer partitions are used to index EIS A000110 – set partitions.

3.1 Set Partitions and Bell polynomials

The following table displays the isomorphisms between the integer partitions of $p(4)$, the set partitions of b_4 and Bell polynomial Y_4. Equation 6 is an analytic functor that maps the set partitions into the Bell polynomials allowing the Bell polynomials to be derived directly from the combinatoric structure of set partitions. See Table 2.

Integer Partitions	**Set Partitions**	**Bell polynomials**
1+1+1+1	{{1},{2},{3},{4}}	$g'(z)^4\, f^{(4)}(g(z))$
2+1+1	{{1,2},{3},{4}}, {{1,3},{2},{4}}, {{1},{2,3},{4}}, {{1,4},{2},{3}}, {{1},{2,4},{3}}, {{1},{2},{3,4}}	$6g'(z)^2\, g''(z)\, f'''(g(z))$
2+2	{{1,2},{3,4}}, {{1,3},{2,4}}, {{1,4},{2,3}}	$3g''(z)^2\, f''(g(z))$
3+1	{{1,2,3},{4}}, {{1,2,4},{3}} {{1,3,4},{2}}, {{1},{2,3,4}}	$4g'(z)\, g'''(z)\, f''(g(z))$
4	{{1,2,3,4}}	$g^{(4)}(z)\, f'(g(z))$

Table 2: Isomorphisms of the Bell polynomial Y_4

The partition function or integer partitions partition unlabeled items while the set partitions partition labeled items. Bell polynomials use the integer partitions as their summation index in Equation 6 because the analytic functor that maps the set partitions into Bell polynomials "forgets" the labels of the items. The integer partitions only retain the distinctive topological structure of the set partitions. Expressing the symmetries of the different topological structures allows the original set partitions to be recovered.

Figure 1 displays the combinatoric functors associated with the pointing operator Θ that are isomorphic to differentiation[6]. This is a visualization of the fact that b_{n+1} can be generated from b_n by using the pointing operator to step through the structure of a set partition and inserting a new item at the end of each set partition's partition or adding a new partition with a single item.

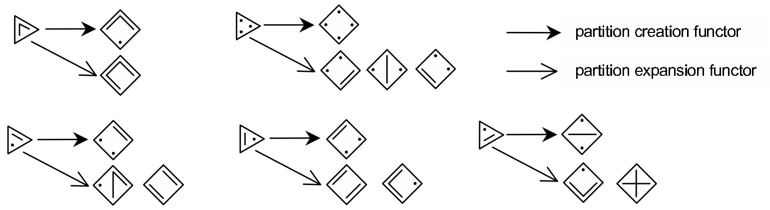

Figure 1: Constructing $Bell(4)$ from $Bell(3)$

3.2 Set Partitions and Total Partitions

The combinatoric structure total partitions was discovered by Ernst Schroeder while investigating the number of ways that the letters can be parenthesized in his 1870 paper, *Vier combinatorische Probleme*[10]. Schroeder wrote the first paper on iterated functions, *Über iterirte Functionen*[11] just one year later, in what many consider the first work on dynamical systems.

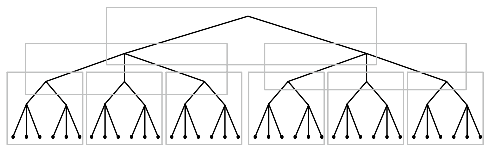

Figure 2: A total partition decomposed into set partitions

Theorem 2 *Recursive set partitions are isomorphic to total partitions.*

Proof. Consider the recursive definition of Equation 7. Given that Equation 6 is isomorphic to set partitions and the ways n items can be parenthesised, Equation 7 can be seen to be isomorphic to the ways n items can be recursively partitioned or parenthesized which is the total partitions. ■

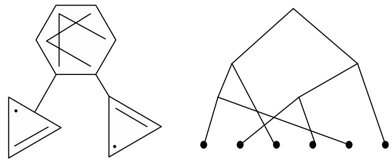

Figure 3: A total partition as a Schroeder diagram and a tree

A consequence of this is that because set partitions are represented by Bell number diagrams, total partitions can be represented by recursive Bell number diagrams, which will be referred to as *Schroeder diagrams*. Figure 3 shows an example. Partitions with one or two items are ignored in the recursion of Bell number diagrams since $hier_1 = 1$ and $hier_2 = 1$. Partitions with three or more items generate further recursive Bell number diagrams as the leaves of the original Bell number diagram, creating a rooted tree of Bell number diagrams.

A caveat to the isomorphism of total partitions and "recursive set partitions" is the first term of their sequences is different; for total partitions $hier_0 = 0$, while for set partitions $b_0 = 1$. A second issue is that for the set partitions of a Bell number, the single set partition with all items in one partition must be illegal to prevent the occurrence of trees of infinite depth. This problem is actually beneficial because this single set partition is isomorphic to a hierarchy of height 1. This provides a mechanism allowing the set partitions, which are trees of depth 2, to decompose total partitions of depth 1. Figure 4 shows the shows the illegal diagrams for b_3, b_4 and b_5.

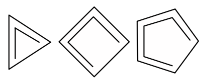

Figure 4: Illegal Schroeder Diagrams

The combinatoric structure total partitions has a related structure unlabeled hierarchies, which is a "recursive integer partition" and serves as an index to total partitions in the same way integer partitions index the set partitions. Due to the symmetries of the total partitions in the Schroeder summations, $D^6 f^n(0)$ only has 33 terms to be evaluated that correspond to the unlabeled hierarchies, instead of the 2752 terms of total partitions.

An analytic functor that is a recursive version of Equation 6 is constructed that maps total partitions directly into the Schroeder summations that constitute the derivatives of iterated functions. See Table 2.

Figure 5 provides a visual representation of how recursive set partitions gives rise to total partitions by using recursive Bell number diagrams and the pointing operator to construct the total partitions of 4 items from total partitions of 3 items. Figure 5 is a recursive version of Figure 1.

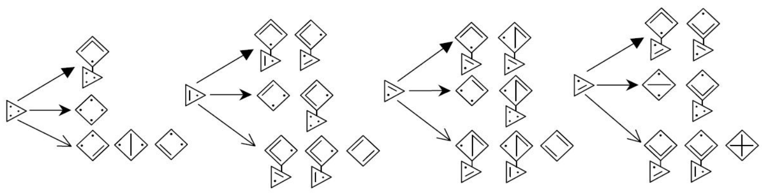

Figure 5: Constructing $hier_4$ from $hier_3$

4 Classification of Fixed Points

Since this work lies within the domain of complex dynamics, one method to refute it is to show that it is inconsistent with complex dynamics.

5 Conclusion

Iterated functions are known for their complexity, but they provide a simple tool for examining the combinatorial structure of Bell polynomials. The question, what is the combinatorial structure of iterated functions, is raised in *Continuous iteration of dynamical maps*[1]. The answer is that total partitions are the combinatorial structure of iterated functions. *Continuous iteration of dynamical maps* uses Bell polynomials in the form of Bell matrices to continuously iterate functions. The author will be take up the issue of continuously iterated functions in a later paper.

The Mathematica software files Iterate.m and SchroederSummations.nb used to compute the first eight derivatives of an iterated function can be found at

http://tetration.org/Resources/Files/Mathematica.

The recent successes of Britto-Cachazo-Feng-Witten recursion[2] and the validation of the energy spectrum of the Hofstadter Butterfly[8] show instances where physics can best be modeled using recursion.

Since physics itself is modeled by recursion, this implies that either the recursion of physics and the recursion of specific physical systems are either the same or more than one recursive relationship is in effect.

The connection between set partitions and Bell polynomials is well known, but Riordan's book Combinatorial Identities also notes an association with the combinatorial structure known as Schroeder's Fourth Problem[10], total partitions[12], or hierarchies[5].

While it can be argued that Riordan implies that total partitions are the combinatoric structure underlying Bell polynomials, a formal proof will be presented here for the case of iterated functions. Iterated functions as an arbitrary number of compositions of a single function are shown to be particularly useful in demonstrating the connection between Bell polynomials and total partitions. It will be shown that total partitions are a recursive version of set partitions. The first four Bell polynomials of iterated functions are derived to add clarity to the general proofs about Bell polynomials of iterated functions. They will also explicitly express the related total partition number in terms of coefficients and in the relationships between the summations used.

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(Concerned with sequences A000041, A000110, A000311, A000669.)